

N^3 -behavior from 5D Yang-Mills theory

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Abstract

In this note we derive N^3 -behavior at large 't Hooft coupling for the free energy of 5D maximally supersymmetric Yang-Mills theory on S^5 . We also consider a Z_k quiver of this model, as well as a model with M hypermultiplets in the fundamental representation. We compare the results to the supergravity description and comment on their relation.

1 Introduction

Recently there has been renewed interest in 6-dimensional $(2, 0)$ superconformal theories. These theories do not admit a standard Lagrangian description, making it difficult to study them directly. Much of our information about these theories comes through the AdS/CFT correspondence, where the $(2, 0)$ theories are conjectured to be dual to M -theory (or supergravity) on an $AdS_7 \times S^4$ background. In particular, the supergravity dual reveals a mysterious N^3 dependence for the free-energy for the $(2, 0)$ theories [1, 2].

The $(2, 0)$ theory lives on the boundary of AdS_7 , which in its Lorentzian version can be chosen to be $R \times S^5$. However, the Euclidean counterpart to this boundary can have R compactified to S^1 . For the dual theory, compactifying one Euclidean direction to S^1 reduces the $(2, 0)$ theory to 5-dimensional maximally supersymmetric Yang-Mills (SYM) theory. Recently it has been suggested that the maximal 5D SYM theory contains all degrees of freedom of the $(2, 0)$ theory, where the Kaluza-Klein states from the S^1 are mapped to the instantons of the 5D theory [3, 4] (see also [5]). Since the N^3 behavior remains in the supergravity dual after compactification, one might expect to find some indication of this N^3 behavior in 5D SYM.

In this note we consider the recent calculations of the $\mathcal{N} = 1$ SYM partition function on S^5 . We show in the case where there is one adjoint hypermultiplet, which in the large radius limit has an enhanced $\mathcal{N} = 2$ supersymmetry¹, that the free energy scales as N^3 , agreeing with the expectation from supergravity. However, if we take the suggested identification of g_{YM}^2 with the radius of S^1 we find a mismatch with the N^3 coefficient. We also consider a Z_k quiver of the $\mathcal{N} = 2$ model which also exhibits N^3 behavior. In order for the k dependence of the matrix model calculation to agree with the corresponding supergravity calculation, there should be an additional factor of k in the identification of the S^1 radius and g_{YM}^2 . Finally, we consider the free-energy for $\mathcal{N} = 1$ models with M hypermultiplets in the fundamental representation. In this case the free-energy scales as N^2 for $M \leq 2N$ in the strong-coupling limit. If $M > 2N$ then the strong-coupling limit is destabilized.

In a related paper [6], it was shown that N^3 behavior can arise from a different formulation of SYM on S^5 . Here, localization reduces the partition function to one almost identical to a Chern-Simons partition function, where it was previously demonstrated to have N^3 behavior in the strong-coupling limit [7–9].

The rest of this note is organized as follows: in section 2 we briefly review the structure of the partition function for 5D SYM on S^5 . In section 3 we analyze the large N -behavior at large 't Hooft coupling of the corresponding matrix model for the $\mathcal{N} = 2$, its Z_k quiver and models with M hypermultiplets in the fundamental representation. In section 4 we review the supergravity analysis for $AdS_7 \times S^4$. In section 5 we compare the gauge theory result with the supergravity result and comment on the numerical mismatch.

¹We will refer to this as an $\mathcal{N} = 2$ model, even though it is not clear that the theory on S^5 actually preserves 16 supersymmetries.

2 5D supersymmetric Yang-Mills theory on S^5

In this section we briefly review the status of 5D SYM theory on S^5 . On \mathbb{R}^5 the $\mathcal{N} = 1$ SYM theory is invariant under 8 supercharges while $\mathcal{N} = 2$ SYM is maximally supersymmetric and is invariant under 16 supercharges. The matter content of the $\mathcal{N} = 2$ theory contains an $\mathcal{N} = 1$ vector multiplet plus an $\mathcal{N} = 1$ hypermultiplet in the adjoint representation. Recently in [10], $\mathcal{N} = 1$ supersymmetric Yang-Mills with hypermultiplets has been constructed on S^5 . Since 5D SYM theory is not superconformal, there is no canonical way to put it on S^5 . However we can think of $\mathcal{N} = 1$ Yang-Mills theory with hypermultiplets as a deformation of the flat theory controlled by the parameter r , where r is the radius of S^5 . Once the limit $r \rightarrow \infty$ is taken, all formulae consistently collapse to the flat case. Thus, $\mathcal{N} = 1$ SYM with a hypermultiplet in the adjoint representation on S^5 produces a deformation of flat $\mathcal{N} = 2$ SYM, where 8 supercharges are explicitly preserved.

The partition function is obtained using localization. For the localization to work on S^5 one needs at least $\mathcal{N} = 1$ supersymmetry. Based on the earlier papers [11] and [10], the localization for $\mathcal{N} = 1$ SYM was analyzed in [12]. There it was argued that the full partition function for $\mathcal{N} = 1$ SYM theory with a hypermultiplet in representation R has the following form

$$Z = \int_{\text{Cartan}} [d\phi] e^{-\frac{4\pi^3 r}{g_{YM}^2} \text{Tr}(\phi^2)} \det_{\text{Ad}} \left(\sin(i\pi\phi) e^{\frac{1}{2}f(i\phi)} \right) \times \det_R \left((\cos(i\pi\phi))^{\frac{1}{4}} e^{-\frac{1}{4}f(\frac{1}{2}-i\phi) - \frac{1}{4}f(\frac{1}{2}+i\phi)} \right) + \mathcal{O}(e^{-\frac{16\pi^3 r}{g_{YM}^2}}), \quad (2.1)$$

where g_{YM} is Yang-Mills coupling constant. For the case of a hypermultiplet in the adjoint representation the answer can be rewritten in the following form

$$Z = \int_{\text{Cartan}} [d\phi] e^{-\frac{4\pi^3 r}{g_{YM}^2} \text{Tr}(\phi^2)} \prod_{\beta} (\sin(\pi\langle\beta, i\phi\rangle) (\cos(\pi\langle\beta, i\phi\rangle))^{\frac{1}{4}} \times e^{\frac{1}{2}f(\langle\beta, i\phi\rangle) - \frac{1}{4}f(\frac{1}{2}-\langle\beta, i\phi\rangle) - \frac{1}{4}f(\frac{1}{2}+\langle\beta, i\phi\rangle)} + \mathcal{O}(e^{-\frac{16\pi^3 r}{g_{YM}^2}}), \quad (2.2)$$

where β are the roots and r is the radius of S^5 . Here the function $f(x)$ is given by the following expression

$$f(y) = \frac{i\pi y^3}{3} + y^2 \log(1 - e^{-2\pi i y}) + \frac{iy}{\pi} \text{Li}_2(e^{-2\pi i y}) + \frac{1}{2\pi^2} \text{Li}_3(e^{-2\pi i y}) - \frac{\zeta(3)}{2\pi^2}. \quad (2.3)$$

A very important property of (2.1) is that $\det_R(\cdots) = \sqrt{\det_R(\cdots) \det_{\bar{R}}(\cdots)}$ (see [12] for further explanation). The matrix models in (2.1) and (2.2) correspond to the full perturbative partition functions (i.e. localization around the trivial connection). All corrections coming from instantons are contributing in with overall factors $\exp(-\frac{16\pi^3 r}{g_{YM}^2})$, as was argued in [12]. If we introduce the t' Hooft coupling constant

$$\lambda = \frac{g_{YM}^2 N}{r},$$

and consider the large N -limit of the partition function (2.2) while keeping λ fixed then only the matrix integral (2.2) contributes to the leading large N behavior. The instanton contributions are exponentially suppressed in the large N -limit with fixed t' Hooft coupling.

In [6] the authors claim that one can construct an $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on S^5 that preserves 16 supercharges. The model they consider belongs to a class of $\mathcal{N} = 1$ theories with hypermultiplets which admit a two parameter deformation controlled by the radius r and a real parameter Δ , very much in the spirit of the 3D story [13], although in the 5d case we believe that the reality properties of the one-loop determinants coming from the hypermultiplets need to be checked for generic values of Δ . Nevertheless, with a single adjoint hypermultiplet, $\Delta = 1$ corresponds to the $\mathcal{N} = 2$ model studied in [6], while $\Delta = 1/2$ corresponds to the model studied here. As far as we can see, the correct deformation of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory associated to S^5 remains an open problem and requires further study. Our main goal is to study the matrix models (2.1) and (2.2), but in the last section we will comment on the N^3 behavior for generic values of Δ .

3 N^3 -behavior from the matrix model

In this section we analyze the large N -behavior of the matrix models in (2.1) and (2.2). We explicitly find N^3 scaling for the free-energy at the large t' Hooft coupling for the case of an adjoint hypermultiplet and Z_k quiver theory. We also consider the matrix model obtained from the minimal $\mathcal{N} = 1$ theory with M hypermultiplets in the fundamental representation. Here we show that the free-energy scales as N^2 for $M \leq 2N$. If $M > 2N$ then the matrix model destabilizes in the strong-coupling limit.

We start with the matrix model (2.2) rewritten in terms of ϕ eigenvalues

$$Z \sim \int \prod_{i=1}^N d\phi_i \exp \left(-\frac{4\pi^3 r}{g_{YM}^2} \sum_i \phi_i^2 + \sum_{j \neq i} \sum_i \left[\log [\sinh(\pi(\phi_i - \phi_j))] + \frac{1}{4} \log [\cosh(\pi(\phi_i - \phi_j))] \right. \right. \\ \left. \left. + \frac{1}{2} f(i(\phi_i - \phi_j)) - \frac{1}{4} f\left(\frac{1}{2} + i(\phi_i - \phi_j)\right) - \frac{1}{4} f\left(\frac{1}{2} - i(\phi_i - \phi_j)\right) \right] \right). \quad (3.1)$$

The derivative of the function $f(y)$ has the remarkably simple form,

$$\frac{df(y)}{dy} = \pi y^2 \cot(\pi y). \quad (3.2)$$

Using this and some simple trigonometric identities we can derive the saddle point equation for (3.1),

$$\frac{8\pi^3 N}{\lambda} \phi_i = \pi \sum_{j \neq i} \left[(2 - (\phi_i - \phi_j)^2) \coth(\pi(\phi_i - \phi_j)) + \left(\frac{1}{4} + (\phi_i - \phi_j)^2 \right) \tanh(\pi(\phi_i - \phi_j)) \right], \quad (3.3)$$

where we have introduced the t' Hooft coupling constant $\lambda = g_{YM}^2 N/r$. In the strong coupling limit $\lambda \rightarrow \infty$ the eigenvalues are pushed apart and the partition function (3.1) and equation of motion (3.3) can be approximated as

$$Z \sim \int \prod_i d\phi_i e^{-\frac{4\pi^3 N}{\lambda} \sum_i \phi_i^2 + \frac{9\pi}{8} \sum_{j \neq i} \sum_i |\phi_i - \phi_j|} \quad (3.4)$$

and

$$\frac{8\pi^2 N}{\lambda} \phi_i = \frac{9}{4} \sum_{j \neq i} \text{sign}(\phi_i - \phi_j), \quad (3.5)$$

respectively. Assuming that the eigenvalues ϕ_i are ordered, we get the solution

$$\phi_i = \frac{9\lambda}{32\pi^2 N} (2i - N). \quad (3.6)$$

Taking the limit $N \rightarrow \infty$ and substituting the saddle point solution (3.6) back into (3.4), we find the free-energy,

$$F \equiv -\log Z \approx -\frac{27}{256} \frac{g_{YM}^2 N^3}{\pi r}, \quad (3.7)$$

where we used the approximations

$$\sum_{i=1}^N (2i - N)^2 \approx \frac{1}{3} N^3, \quad \sum_{j \neq i} \sum_{i=1}^N |i - j| \approx \frac{1}{3} N^3. \quad (3.8)$$

A related theory to the $\mathcal{N} = 2$ model is a Z_k quiver, where the $SU(N)$ gauge group is broken to $SU(N/k)^k$ and with the hypermultiplets in the bifundamental representations, $(N/k, \overline{N/k}, 1, \dots, 1)$, $(1, N/k, \overline{N/k}, 1, \dots)$, *etc.*. The N eigenvalues that appear in (3.3) can be split into k groups of N/k , $\psi_i^{(r)}$, where $r = 1, \dots, k$ and $i = 1, \dots, N/k$. The equation of motion from the resulting matrix model (2.1) is then

$$\begin{aligned} \frac{8\pi^3 N}{\lambda} \psi_i^{(r)} = & \pi \left[\sum_{j \neq i} \left(2 - (\psi_i^{(r)} - \psi_j^{(r)})^2 \right) \coth(\pi(\psi_i^{(r)} - \psi_j^{(r)})) \right. \\ & + \sum_j \left[\frac{1}{2} \left(\frac{1}{4} + (\psi_i^{(r)} - \psi_j^{(r+1)})^2 \right) \tanh(\pi(\psi_i^{(r)} - \psi_j^{(r+1)})) \right. \\ & \left. \left. + \frac{1}{2} \left(\frac{1}{4} + (\psi_i^{(r)} - \psi_j^{(r-1)})^2 \right) \tanh(\pi(\psi_i^{(r)} - \psi_j^{(r-1)})) \right] \right]. \quad (3.9) \end{aligned}$$

This has a solution where $\psi_i^{(r)} = \psi_i^{(s)}$, in which case (3.9) takes the same form as (3.3), except with N replaced by N/k . Thus, in the strong-coupling limit we have

$$\phi_i = \frac{9g_{YM}^2}{32\pi^2 r} (2i - N/k), \quad (3.10)$$

with free-energy

$$F \approx -k \frac{27}{256} \frac{g_{YM}^2 N^3}{\pi k^3 r} = -\frac{27}{256} \frac{g_{YM}^2 N^3}{\pi k^2 r}. \quad (3.11)$$

In these models, the N^3 behavior arises from the long-range linear repulsive potential between the eigenvalues. In fact, any matrix model with such a potential will give N^3 behavior since it will spread the eigenvalues over a range of order N . However, a generic $\mathcal{N} = 1$ model will not have such a potential.

For example, suppose we consider M hypermultiplets in the fundamental and anti-fundamental representations. In this case the eigenvalue equation for matrix model (2.1) becomes

$$\begin{aligned} \frac{8\pi^3 N}{\lambda} \phi_i = \pi \left(\sum_{j \neq i} [(2 - (\phi_i - \phi_j)^2) \coth(\pi(\phi_i - \phi_j))] \right. \\ \left. + \frac{M}{2} \left(\frac{1}{4} + \phi_i^2 \right) \tanh(\pi \phi_i) \right). \end{aligned} \quad (3.12)$$

Since N^3 behavior requires well separated eigenvalues, let us assume that they are, in which case we can approximate (3.12) as

$$\begin{aligned} \frac{8\pi^3 N}{\lambda} \phi_i = \pi \left(\sum_{j \neq i} [(2 - (\phi_i - \phi_j)^2) \operatorname{sgn}(\phi_i - \phi_j)] \right. \\ \left. + \frac{M}{2} \left(\frac{1}{4} + \phi_i^2 \right) \operatorname{sgn}(\phi_i) \right). \end{aligned} \quad (3.13)$$

Taking the limit $N \rightarrow \infty$ and defining $x = i/N - 1/2$, we can rewrite (3.13) as

$$\begin{aligned} \frac{8\pi^3 N}{\lambda} \phi(x) = N\pi \left[4x - 2 \left(x\phi^2(x) - 2\phi(x) \left(\Phi(x) - \Phi\left(\frac{1}{2}\right) \right) + \Phi_2(x) \right) \right. \\ \left. + \frac{M}{2N} \left(\frac{1}{4} + \phi^2(x) \right) \operatorname{sgn}(x) \right], \end{aligned} \quad (3.14)$$

where $\Phi(x) = \int_0^x \phi dx$ and $\Phi_2(x) = \int_0^x \phi^2 dx$. In (3.14) we have assumed that $\phi(x)$ is a monotonic increasing odd function, although not necessarily continuous.

If we now take an x derivative on both sides of (3.14), we end up with the equation

$$\begin{aligned} \frac{8\pi^3}{\lambda} \phi'(x) &= 4\pi \left[1 - (x - b)\phi(x)\phi'(x) + \phi'(x) \left(\Phi(x) - \Phi\left(\frac{1}{2}\right) \right) \right], \quad x > 0 \\ \frac{8\pi^3}{\lambda} \phi'(x) &= 4\pi \left[1 - (x + b)\phi(x)\phi'(x) + \phi'(x) \left(\Phi(x) - \Phi\left(\frac{1}{2}\right) \right) \right], \quad x < 0 \end{aligned} \quad (3.15)$$

where $b = M/(4N)$. Dividing by $\phi'(x)$ and taking one more x derivative, we arrive at

$$\begin{aligned} 0 &= -\frac{\phi''}{(\phi')^2} - (x - b)\phi', \quad x > 0 \\ 0 &= -\frac{\phi''}{(\phi')^2} - (x + b)\phi', \quad x < 0 \end{aligned} \quad (3.16)$$

which has the solution

$$\begin{aligned} \phi(x) &= \operatorname{arcsinh} \left(\frac{x - b}{c} \right) + C, \quad 0 < x \leq \frac{1}{2} \\ \phi(x) &= \operatorname{arcsinh} \left(\frac{x + b}{c} \right) - C, \quad -\frac{1}{2} \leq x < 0, \end{aligned} \quad (3.17)$$

where c and C are yet to be determined constants. Substituting this solution back into (3.15) leads to the relation

$$\frac{4\pi^2}{\lambda} = \sqrt{4c^2 + (1-2b)^2} - (1-2b) \left(\operatorname{arcsinh} \left(\frac{1-2b}{2c} \right) + C \right). \quad (3.18)$$

If we substitute (3.17) into (3.14), then using (3.18) we find the solutions for C

$$C = \operatorname{arcsinh} \frac{b}{c} \pm \sqrt{\frac{c^2}{b^2} + \frac{5}{4}} - \sqrt{\frac{c^2}{b^2} + 1}, \quad (3.19)$$

where only the solution with the $+$ sign is consistent with the monotonicity of $\phi(x)$.

If $b < 1/2$ then there exists a positive real value of c that satisfies (3.18) and (3.19), even if $\lambda \rightarrow \infty$. (3.19) shows that $\phi(x)$ is positive (negative) for positive (negative) x and has a finite jump at $x = 0$. Since c is nonzero, (3.17) and (3.19) show that the eigenvalues are distributed over a finite range and thus the approximation in (3.13) is not valid. Nonetheless, it is still true that the eigenvalues are only over a finite extent, hence the free-energy can only scale as N^2 since all ϕ_i and $\phi_i - \phi_j$ are finite in the large N limit.

If $b = 1/2$, which corresponds to $M = 2N$, then (3.18) gives $c = 2\pi^2/\lambda$. In the strong coupling limit, $c \rightarrow 0$ and we can approximate $\phi(x)$ as

$$\begin{aligned} \phi(x) &\approx \log \frac{1}{1-2x} + \frac{\sqrt{5}-2}{2} & 0 < x < \frac{1}{2}, \\ \phi(x) &\approx -\log \frac{1}{1+2x} - \frac{\sqrt{5}-2}{2} & -\frac{1}{2} < x < 0 \end{aligned} \quad (3.20)$$

away from the boundary points $x = \pm \frac{1}{2}$ and

$$\phi(\pm \frac{1}{2}) \approx \pm \left(\log \frac{1}{c} + \frac{\sqrt{5}-2}{2} \right) \quad (3.21)$$

at these points. Hence, the eigenvalues spread out over an infinite distance as $c \rightarrow 0$. However, a finite fraction are within a finite region, for example, half the eigenvalues lie between $\pm(\log 2 + \frac{\sqrt{5}-2}{2})$. Thus, the approximation in (3.13) is not completely valid. Using it anyway, one can easily check that the free-energy that gives the equation of motion in (3.12) scales as N^2 with the eigenvalue distribution in (3.20).

Finally, if $b > 1/2$ then (3.18) has no real solution for c in the strong coupling limit, suggesting that the eigenvalue distribution destabilizes.

4 Comparison with supergravity on $AdS_7 \times S^4$

We now compare our results in the previous section to the supergravity result on $AdS_7 \times S^4$ where the AdS_7 boundary is $S^1 \times S^5$. The radius of AdS_7 is ℓ , while that of the S^4 is $\ell/2$, where $\ell = 2\ell_{pl}(\pi N)^{1/3}$. The AdS_7 metric can then be written in the form

$$ds^2 = \ell^2 (\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_5^2), \quad (4.1)$$

where $d\Omega_5^2$ is the round metric for the unit 5-sphere and $\tau \equiv \tau + 2\pi R_6/r$. R_6 and r are the radii of S^1 and S^5 on the boundary.

According to the AdS/CFT correspondence, the supergravity classical action equals the free-energy of the boundary field theory. The action itself is divergent so it needs to be regulated by adding counterterms [14–17]. The full action then has the form

$$I_{AdS} = I_{\text{bulk}} + I_{\text{surface}} + I_{\text{ct}} , \quad (4.2)$$

where

$$I_{\text{bulk}} = -\frac{1}{16\pi G_N} \text{Vol}(S^4) \int d^7x \sqrt{g} (R - 2\Lambda) \quad (4.3)$$

is the action in the bulk, I_{surf} is the surface contribution and I_{ct} contains counterterms written only in terms of the boundary metric and which cancel off divergences in I_{bulk} . We use the convention in [18] for G_N , $G_N = 1/(16\pi^7 \ell_{pl}^9)$. Using that

$$R - 2\Lambda = -\frac{12}{\ell^2} , \quad (4.4)$$

we have

$$I_{\text{bulk}} = -\frac{1}{256\pi^8 \ell_{pl}^9} \left(\frac{\pi^2 \ell^4}{6} \right) \frac{2\pi R_6}{r} \pi^3 (-12\ell^5) \int_0^{\rho_0} \cosh \rho \sinh^5 \rho d\rho = \frac{4\pi R_6}{3r} N^3 \sinh^6 \rho_0 . \quad (4.5)$$

In the limit that $\rho_0 \rightarrow \infty$ the integral is divergent and corresponds to a UV divergence for the boundary theory. In terms of an ϵ expansion of the boundary theory, we make the identification $\epsilon = e^{-\rho_0}$, which then gives

$$\sinh^6 \rho_0 = \frac{1}{64} \epsilon^{-6} - \frac{3}{32} \epsilon^{-4} + \frac{15}{64} \epsilon^{-2} - \frac{5}{16} + \mathcal{O}(\epsilon^2) . \quad (4.6)$$

The surface term contributes to the divergent pieces, but not the finite part of (4.5), while the effect of the counterterm is to cancel off the divergent pieces. Hence, we find [15]

$$I_{AdS} = -\frac{5\pi R_6}{12r} N^3 . \quad (4.7)$$

The supergravity dual of a $(2, 0)$ Z_k quiver theory is expected to be $AdS_7 \times S^4/Z_k$, where S^4/Z_k is a Z_k orbifold of S^4 [19]. The only change in the preceding calculation is to replace $\text{Vol}(S^4)$ with the $\text{Vol}(S^4/Z_k) = \text{Vol}(S^4)/k$. Hence, the regularized action is

$$I_{AdS} = -\frac{5\pi R_6}{12kr} N^3 . \quad (4.8)$$

5 Discussion

We can now compare the gauge theory result in (3.7) with the supergravity result in (4.7). The good news is that they both have N^3 behavior. To compare the numerical factors we need a relation between g_{YM}^2 and R_6 . As suggested in [3, 4] we can identify the KK states on $S^1 \times \mathbb{R}^5$ with the instanton particles on \mathbb{R}^5 and arrive at the following identification

$$R_6 = \frac{g_{YM}^2}{8\pi^2} . \quad (5.1)$$

Using this relation the supergravity result becomes

$$I_{AdS} = -\frac{5 g_{YM}^2}{96\pi r} N^3 , \quad (5.2)$$

which is off by a factor of 160/81 from the gauge theory calculation (3.7). For the quiver theory, if we use (5.1) then the power of k in (3.11) does not match with (4.8). This suggests that the identification between the S^1 radius and g_{YM}^2 should be

$$R_6 = \frac{g_{YM}^2}{8\pi^2 k} . \quad (5.3)$$

If we take the matrix model suggested in [6] (which only has the sine factors in the determinant), then as pointed out in [6] one can evaluate the integral directly [20], where one finds a factor of $N(N^2 - 1)$ in the free-energy. Alternatively, one can use the analysis from section 3 to find the leading N^3 factor. The resulting free-energy is given by (3.7) multiplied by a factor of 64/81. This still has the N^3 -behavior, but the numerical mismatch with (4.7) remains. If we consider the more general models in [6] parameterized by Δ , then the analysis in section 3 gives

$$F \equiv -\log Z \approx -\frac{(2 - \Delta)^2(1 + \Delta)^2}{48} \frac{g_{YM}^2 N^3}{\pi r} , \quad -1 < \Delta < 2 , \quad (5.4)$$

which is minimized for $\Delta = 1/2$. If Δ is outside the bounds in (5.4) then there will be long-range attraction between the eigenvalues which cannot lead to N^3 behavior [21].

A possible explanation for the numerical mismatch is that we are looking at the wrong 5D SYM theory on S^5 . Since the theory is not superconformal, there is no canonical way to put it on the sphere. Moreover, we can add to the 5D Yang-Mills action a supersymmetric Chern-Simons term, thus modifying the numerics of the matrix model. Another possibility is that the relation between R_6 and g_{YM} on $S^1 \times S^5$ differs from the one suggested in (3) by [3, 4].

Finally, the finite part of (4.5) is actually scheme dependent, as one can add a local counterterm to the boundary which is proportional to the conformal anomaly $[16]^2$. Choosing a different scheme could then change (4.7). In fact, since we are really considering a 5-dimensional theory, it may be more appropriate to consider supergravity backgrounds sourced by D4 branes [22, 23]. In this case, one could also allow local counterterms that are covariant in five dimensions but not in six [24].

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